Satz 2: Die axialsymmetrischen, stationären Lösungen der Feldgleichungen für druckfreie Materie sind auf umkehrbar eindeutige Weise den axialsymmetrischen, statischen Vakuumlösungen zugeordnet.

Materie mit Druck

Im allgemeinen Fall ($\mu \neq 0 \neq p$) sind bisher keine physikalisch brauchbaren Lösungen angegeben worden, Lösungen also, die zu einer räumlich begrenzten Massenverteilung gehören. Das interessanteste, zweifellos schwierige Problem dürfte sein, eine innere Lösung zu finden, die sich an das von KERR angegebene Außenfeld anschließt. Auf der Grenzfläche $p=0$, die zugleich eine Fläche $U = \text{const}$ ist, verlangen die Anschlußbedingungen von LICHNEROWICZ die Stetigkeit der metrischen Koeffizienten und ihrer ersten Normalableitungen. Auch hier, wie im Fall von äußeren Lösungen, kann nur eine Klärung des Randwertproblems weitere Fortschritte bringen.

Isometry Groups with Surface-Orthogonal Trajectories

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Dedicated to Professor PASCUAL JORDAN *on the occasion of his 65th birthday*

It is shown that the trajectories of an isometry group admit orthogonal surfaces if the sub-group of stability leaves no vector in the tangent space of the trajectories fixed. A necessary and sufficient condition is given that the trajectories of an ABELian group admit orthogonal surfaces.

In spacetimes which admit an ABELian G^2 of isometries, the trajectories admit orthogonal 2-surfaces if a timelike congruence exists with the following properties: the curves lie in the trajectories and are invariant under G^2; ω^a and $\ddot{u}^a$ are linearly independent and orthogonal to the trajectories.

Many of the known exact solutions of EINSTEIN's field equations are spacetimes admitting isometry groups, because symmetries simplify the problem of solving the equations. A considerable simplification can be achieved in cases where the trajectories of the groups admit orthogonal surfaces. Furthermore CARTER [1] has recently pointed out a global significance of orthogonal surfaces.

A theorem of § 1 shows that a certain property of the sub-group of stability or isotropy-group is sufficient for the existence of orthogonal surfaces. A second theorem shows that another property of the sub-group of stability implies the trajectories to be conformally mapped by the orthogonal surfaces. This fact makes possible a further strong simplification of the metric. The assumptions of both theorems are satisfied if the isometry group is the maximal group in the trajectories. An example for this case is a 3-dimensional group with 2-dimensional trajectories. The application of this reasoning to spherically symmetric space times yields a simple derivation of the line element.

We can prove these theorems without any calculation. We simply use the properties of an isometry group, for example that a geodesic is mapped into a geodesic. This consideration is superior to the usual method (integration of the KILLING equation with a given representation of the LIE algebra of the group as infinitesimal transformations on a manifold). Additionally, we obtain not only the line element in its simplest form, but we even understand geometrically how the properties of the group imply these simplifications.

In § 2 the results of § 1 are applied to ABELian isometry groups. We will give necessary and sufficient conditions that the trajectories of the group admit orthogonal surfaces. In this context a discrete isometry — a reflection — is particularly important.

In § 3 we consider spacetimes with an ABELian group G^2 of isometries. The stationary field of a rotating body has this symmetry. Concerning the integration of the field equations it is important to know whether the trajectories admit orthogonal surfaces or not. We show that the following properties are necessary conditions: the timelike congruence

[1] B. CARTER, Preprint Cambridge 1966.

representing the matter is rigid; the vorticity vector ω^a as well as the acceleration vector $\dot{u}^a$ is orthogonal to the trajectories. These conditions are also sufficient, if ω^a and $\dot{u}^a$ are linearly independent.

This way the problem of the orthogonal decomposition of the metric is reduced to kinematic properties of the matter. No field equations are used. Working with the reflection results in considerable simplifications of the proofs.

§ 1. Orthogonal Decomposition of Spaces with an Isometry Group

We consider intransitive isometry groups, always assuming that the trajectories are not degenerated (i. e. every trajectory forms a RIEMANNian space itself).

Theorem 1: *Assumptions:* G^r is an r-dimensional — not necessarily connected — LIE group of isometric transformations on a RIEMANNian manifold V^n of any signature. The trajectories T^q of G^r are connected and q-dimensional. The sub-group of stability F_P of the point P leaves no vector of the tangent space $\mathcal{T}_P{}^q$ of $T_P{}^q$ fixed.

Assertion: The trajectories admit orthogonal $(n-q)$-surfaces.

Proof: Let $\mathcal{T}_P{}^\perp$ be the subspace of the tangent space of P containing all vectors orthogonal to $\mathcal{T}_P$, which are not in $\mathcal{T}_P$. First we show that F_P leaves fixed every vector of $\mathcal{T}_P{}^\perp$.

Let us consider the set of all geodesics passing through P with an initial direction in $\mathcal{T}_P{}^\perp$. It forms an $(n-q)$-dimensional surface C. The trajectories passing through two different points of C can have no common point, because in a neighbourhood of P the space is a topological product of C and $T_P{}^q$ (the trajectory passing through P). Let us assume that there are two vectors in $\mathcal{T}_P{}^\perp$, which can be transformed into one another by an element of F_P. Then the two geodesics in these directions are likewise transformed one upon another. Therefore there are different points of C lying in the same trajectory. Since this statement contradicts our previous result we have proved that F_P leaves fixed each vector of $\mathcal{T}_P{}^\perp$.

Furthermore, the whole surface C is kept fixed by F_P, because each point is on a geodesic passing through P with an initial direction in $\mathcal{T}_P{}^\perp$. F_P leaves fixed this direction and therefore the geodesic.

Let Q be any point in C, and let λ be any vector of the tangent space of C in Q. F_P leaves fixed λ, because F_P leaves fixed C. Suppose $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 \in \mathcal{T}_Q$ and $\lambda_2 \in \mathcal{T}_Q{}^\perp$. λ_2 is kept fixed by F_P as we have shown above. λ_1 has to change according to the assumption of the theorem. Therefore λ_1 is zero. That means $\lambda = \lambda_2 \in \mathcal{T}_Q{}^\perp$ is orthogonal to the trajectory possing through Q. Thus in each point C is orthogonal to the trajectories. This proves the theorem.

Theorem 2: *Assumptions:* G^r is an r-dimensional isometry group on a RIEMANNian manifold V^n with q-dimensional trajectories T^q. The trajectories admit $(n-q)$-dimensional orthogonal surfaces $T^\perp$. In the tangent space $\mathcal{T}_P{}^q$ of the trajectory $T_P{}^q$ there is a basis of non-null vectors which has the following property: for any two vectors of the basis there is an element of the group of stability F_P which maps one into the other.

Assertion: The orthogonal surfaces $T^\perp$ map the trajectories conformally upon each other.

Proof: Let T_P and T_Q be two different trajectories, and let P be a point in T_P. We can define a one-to-one mapping f between T_P and T_Q by

$$f\colon\ P \to T_Q \cap T_P^\perp =: f(P). \tag{1}$$

We have $fg = gf$ for all $g \in G^r$ because

$$gf(P) = g(T_Q \cap T_P^\perp) = T_Q \cap T_{g(P)}^\perp = f(g(P)) = fg(P). \tag{2}$$

Let $e_1, e_2, \ldots, e_q$ be a basis of $\mathcal{T}_P{}^q$ such that $e_i = g_i(e_1)$ $i = 2, \ldots, q$ with $g_i \in F_P$ and $e_1{}^2 \neq 0$. Then $f(e_1), \ldots, f(e_q)$ form a basis of $\mathcal{T}_{f(P)}^q$. From (2) we get $f(e_i) = f(g_i(e_1)) = g_i(f(e_1))$. If T_Q is chosen near T_P, $f(e_1)$ has non-vanishing norm. Therefore a basis of unit vectors in $\mathcal{T}_P$ is mapped into a basis of vectors having equal norm in $\mathcal{T}_{f(P)}$. Since the mapping induced by f between the tangent spaces is linear, it is a conformal one. The conform factor has to be constant on T_Q, because we can compose the mapping $\mathcal{T}_{P'} \to \mathcal{T}_{f(P')}$ $(P' \in T_P)$ in the following way:

$$\mathcal{T}_{P'} \xrightarrow{h} \mathcal{T}_P \xrightarrow{f} \mathcal{T}_{f(P)} \xrightarrow{h^{-1}} \mathcal{T}_{h^{-1}f(P)} = \mathcal{T}_{f(P')}\,. \tag{3}$$

There h is an isometry of G^r with $h(P') = P$. Thus the proof is complete.

If we choose suitable coordinates (x^α, x^A) $\alpha = 1, \ldots, q$; $A = q+1, \ldots, n$ $x^A = \text{const}$ describing T^q, $x^\alpha = \text{const}$ describing $T^\perp$, the metric has the form

$$ds^2 = B^2(x^A)\,(g_{\alpha\beta}(x^\alpha)\,dx^\alpha\,dx^\beta) + g_{AB}(x^A)\,dx^A\,dx^B. \tag{4}$$

Using both theorems we can easily prove

Theorem 3: Given a V^n with a G^r of isometries with q-dimensional trajectories, and $r = q(q+1)/2$. Then the trajectories admit orthogonal surfaces, and in suitable coordinates the metric has the form (4).

Proof: The trajectories T^q are spaces of constant curvature, because r is the maximal dimension of an isometry group in a q-dimensional space. A space of constant curvature has the maximal group of stability, and therefore two vectors of the same norm can be mapped one upon another by an element of this group. Therefore no vector in T^q is kept fixed under the whole sub-group of stability. Thus the assumptions of theorem 1 are satisfied, and the trajectories admit orthogonal surfaces.

In a metric vector space of any signature there is always a basis consisting of vectors of equal non-vanishing norm. Two vectors of any pair of vectors with equal norm can be mapped upon each other by an element of the sub-group of stability, since we have maximal isotropy in the trajectories. This is the second assumption of theorem 2, and thus we have proved theorem 3.

If we apply this theorem to the 3-dimensional group of rotations we get the well known normal form of spherically symmetric spacetimes:

$$ds = B^2(r,t)\, d\tau^2(\Theta,\varphi) + d\sigma^2(r,t). \qquad (5)$$

$d\tau^2$ is the definit metric of a surface of constant positive curvature, and $d\sigma^2$ is any indefinite surface.

§ 2. Abelian Isometry Groups

Now we shall derive the necessary and sufficient conditions that the trajectories of an ABELian isometry group admit orthogonal surfaces.

Lemma 1: Let G be a group of transformations on a manifold M^n. Suppose there is a one-to-one mapping s of M^n onto itself which has the following property

$$g\,s\,g = s \quad \text{for all} \quad g \in G. \qquad (6)$$

Then G is an ABELian group.

Proof: If g, $h \in G$ we have from (6)

$$g\,s\,g = s. \qquad (7)$$

By multiplication with f we get

$$f\,g\,s\,g\,f = f\,s\,f = s. \qquad (8)$$

Using (6) for the element $f\,g \in G$ we obtain

$$f\,g\,s\,f\,g = s. \qquad (9)$$

(8) and (9) imply $f\,g = g\,f$.

Lemma 2: Let G^r be a LIE group of transformations acting on a monifold M^n, and s a one-to-one mapping of M^n onto itself, mapping each trajectory of G^r onto itself, and satisfying $g\,s\,g = s$ for each $g \in G^r$. Then we have:

(a) G^r is an ABELian group.

(b) s has a fixed-point in each trajectory T of G^r.

(c) In a neighbourhood of a fixed-point there exists a chart (x^i) such that s has the form

$$s : (x^\alpha, x^A) \to (-x^\alpha, x^A) \quad \begin{array}{l} \alpha = 1, \dots, r, \\ A = r+1, \dots, n. \end{array} \qquad (10)$$

Proof: (a) is shown by lemma 1.

(b): G^r, being ABELian, acts simply transitive on T. Suppose $P_0 \in M^n$, then P_0 and $s(P_0)$ are in the same trajectory. Since G^r acts simply transitive, there exists a mapping $f \in G^r$ with $f(P_0) = s(P_0)$. Therefore $f^{-1}s$ leaves P_0 fixed. Be h an arbitrary element of G^r. Then $s' = h\,s$ satisfies $g\,s'\,g = s'$ for each $g \in G$, for $g\,s'\,g = g\,h\,s\,g = h\,g\,s\,g = h\,s = s'$. $s' := f^{-1}s$ leaves P_0 fixed. Therefore we can assume that s has a fixed-point.

(c): (x^i) be a chart in a neighbourhood of the fixed-point of s. G^r is an ABELian group, therefore we can choose the chart such that the infinitesimal transformations of G^r have the form

$$\xi_\alpha{}^i = \delta_\alpha{}^i \quad (\alpha = 1, \dots, r). \qquad (11)$$

Let a^α be the parameters of G^r, then the finite transformations of G^r are

$$(x^\alpha, x^A) \to (x^\alpha + a^\alpha, x^A). \qquad (12)$$

In the chart (x^i) s is given by $s^i(x^k)$. Rewriting $g\,s\,g = s$ in coordinates we get

$$(s^i(x^k)) = (s^\beta(x^\alpha + a^\alpha, x^A) + a^\beta, s^B(x^\alpha + a^\alpha, x^A)). \qquad (13)$$

Differentiation with respect to a^α and substitution $a^\alpha = 0$ yields

$$0 = s^\beta{}_{,\alpha} + \delta_\alpha{}^\beta, \qquad 0 = s^B{}_{,\alpha}. \qquad (14)$$

By integration of (14) we get

$$s^\beta(x^\alpha, x^A) = -x^\alpha + f^\alpha(x^A), \qquad (15)$$

$$s^B(x^\alpha, x^A) = s^B(x^A). \qquad (16)$$

As s maps each trajectory $x^A = \text{const}$ onto itself, it holds

$$s^B(x^A) = x^B. \qquad (17)$$

Thus the transformation s is given by

$$s : (x^\alpha, x^A) \to (-x^\alpha + f^\alpha(x^A), x^A). \qquad (18)$$

If we take coordinates $(x^{i'})$ defined by

$$x^{\alpha'} = x^\alpha - \tfrac{1}{2} f^\alpha(x^A), \qquad x^{A'} = x^A, \qquad (19)$$

(11) is preserved, and s is given by

$$s : (x^\alpha, x^A) \to (-x^\alpha, x^A). \qquad (20)$$

This completes the proof.

It is possible to perform this calculation in general coordinates. Making use of the differential equations, which the finite transformations of G^r are to satisfy, one obtains the following system that corresponds to (14):

$$s^i{}_{,k}\, \xi_\alpha{}^k(x) + \xi_\alpha{}^i(s(x)) = 0 . \qquad (21)$$

From (21) we could derive again that G^r must be Abelian, for the conditions of integrability of (21) are $[\xi_\alpha, \xi_\beta] = 0$.

Now we can prove the following theorem (an equivalent theorem was independently found by Carter [1]).

Theorem 4: Given a Riemannian manifold V^n which admits an Abelian group G^r of isometries with non-degenerate trajectories. Then the following two statements are equivalent:

(a) The trajectories admit orthogonal $(n-r)$-surfaces.

(b) V^n admits another isometry s which maps each trajectory of G^r onto itself and satisfies $g\, s\, g = s$ for all $g \in G^r$.

Proof: (b → a) According to lemma 2, in a suitable chart (x^i) s is given by

$$s: (x^\alpha, x^A) \to (-x^\alpha, x^A). \qquad (22)$$

The group of stability contains s, and (22) shows that s leaves fixed no vector in the tangent space of the trajectories of G^r: The assumptions of theorem 1 are satisfied, thus the trajectories admit orthogonal surfaces.

(a → b): This is trivial, because in suitable coordinates the metric is given by

$$ds^2 = g_{\alpha\beta}(x^A)\, dx^\alpha\, dx^\beta + g_{AB}(x^A)\, dx^A\, dx^B. \qquad (23)$$

Obviously, s defined by $(x^\alpha, x^A) \to (-x^\alpha, x^A)$ is an isometry, and the relation $g\, s\, g = s$ holds, for g is given by

$$(x^\alpha, x^A) \to (x^\alpha + a^\alpha, x^A) .$$

In the case of V^4, the essential content of theorem 4 has been already stated by Ehlers [2], but no further use has been made of it.

§ 3. Axi-Symmetric Stationary Spacetimes

The stationary field of a rotating body is described in Einstein's theory of gravitation by a spacetime V^4 admitting an Abelian group G^2 of isometries with timelike trajectories. For the integration of the field equations it is important to know under which conditions the trajectories admit orthogonal 2-surfaces.

1966 Papapetrou [3] has shown that the vacuum field equations and the non-local assumption of a subgroup with closed 1-dimensional trajectories are sufficient conditions. Kundt and Trümper [4] have generalized this result to a wider class of field equations including those of a perfect fluid. In § 2 it was shown that the orthogonal surfaces exist if and only if there is a discrete isometry. Now we shall reduce the existence of this isometry to some properties of the kinematics of a timelike congruence.

Theorem 5: Let V^4 be a spacetime admitting an Abelian G^2 of isometries with timelike trajectories and orthogonal 2-surfaces. Then a timelike congruence is necessarily rigid if it is invariant under G^2, and if its tangent vector u^a lies in the trajectories. The vorticity vector ω^a as well as the acceleration vector $\dot{u}^a$ are orthogonal to the trajectories.

Proof: We introduce coordinates such that the Killing vectors ξ^i, η^i have the form $\xi^i = \delta_4{}^i$, $\eta^i = \delta_3{}^i$, and the orthogonal 2-surfaces are given by $x^\alpha = \text{const}$ $(\alpha = 3, 4;\ A = 1, 2)$. According to § 2 the mapping s

$$s: (x^A, x^\alpha) \to (x^A, -x^\alpha) \qquad (24)$$

is an isometry. Any congruence with tangent vector u^a in the trajectories is given by

$$u^a = f(x^i)\, \delta_3{}^a + g(x^i)\, \delta_4{}^a. \qquad (25)$$

The assumption that u^a is invariant under G^2 means $[u^a, \xi^a] = 0$, $[u^a, \eta^a] = 0$, and this yields

$$f_{,\alpha} = g_{,\alpha} = 0 .$$

Thus we have

$$u^a = f(x^A)\, \delta_3{}^a + g(x^A)\, \delta_4{}^a. \qquad (26)$$

From (24) and (26) we see that s transforms u^a into $-u^a$. The kinematical quantities Θ, σ_{ab}, ω_{ab}, ω^a, $\dot{u}^a$ defined by

$$\Theta = u^a{}_{;a} , \qquad (27)$$

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}\, u_{b)} - \Theta\, (g_{ab} + u_a\, u_b) \cdot \tfrac{1}{3} , \qquad (28)$$

[2] J. Ehlers, Dissertation, Universität Hamburg 1957.
[3] A. Papapetrou, Ann. Inst. Henri Poincaré 4, 83 [1966].
[4] W. Kundt and M. Trümper, Z. Physik 192, 419 [1966].

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a}\, u_{b]}\,, \tag{29}$$

$$\dot{u}^a = u^a{}_{;b}\, u^b\,, \tag{30}$$

$$\omega^a = \tfrac{1}{2}\, \eta^{abcd}\, \omega_{bc}\, u_d\,, \tag{31}$$

are transformed in the following way: $\Theta \rightarrow -\Theta$. Since s is an isometry it follows $\Theta = 0$. Consequently $\sigma_{ab} \rightarrow -\sigma_{ab}$, $\omega_{ab} \rightarrow -\omega_{ab}$, $\dot{u}^a \rightarrow \dot{u}^a$, $\omega^a \rightarrow \omega^a$. Thus we get

$$\xi^a\, \dot{u}_a \rightarrow -\xi^a\, \dot{u}_a\,, \qquad \eta^a\, \dot{u}_a \rightarrow -\eta^a\, \dot{u}_a \tag{32}$$

because $\xi^a \rightarrow -\xi^a$, $\eta^a \rightarrow -\eta^a$. The scalars $\xi^a\, \dot{u}_a$ and $\eta^a\, \dot{u}_a$ have to be invariant, therefore they must vanish, and $\dot{u}^a$ is orthogonal to the trajectories. The same consideration yields that ω^a is orthogonal to the trajectories.

Now it remains to show that σ_{ab} vanishes. $\lambda_{(i)}{}^a = \delta_i{}^a$ $(i = 1, \ldots, 4)$ are four linearly independent vectors.

Under s we have

$$\sigma_{ab}\, \lambda_{(i)}{}^a\, \lambda_{(i)}{}^b \rightarrow -\sigma_{ab}\, \lambda_{(i)}{}^a\, \lambda_{(i)}{}^b \quad (i = 1, \ldots, 4) \tag{33}$$

and therefore these scalars must vanish. So we get $\sigma_{ab} = 0$. (Theorem 5 naturally holds for spacelike congruences, too.)

With the additional assumption that ω^a and $\dot{u}^a$ are linearly independent, we can prove a converse of theorem 5.

Theorem 6: Let V^4 be a spacetime admitting an Abelian G^2 of isometries and a timelike congruence which is invariant under G^2, and whose curves lie in the trajectories of G^2. Suppose ω^a, $\dot{u}^a$ are linearly independent and orthogonal to the trajectories.

Then the trajectories admit orthogonal 2-surfaces, and the congruence is rigid.

Proof: We introduce coordinates such that $\eta^a = \delta_3{}^a$, $\xi^a = \delta_4{}^a$ (ξ, η generate G^2). Thus u^a is given by (26). Now we define a mapping s in the following way

$$s\colon (x^A, x^\alpha) \rightarrow (x^A, -x^\alpha)\,. \tag{34}$$

Obviously, s satisfies $g\, s\, g = s$ for all $g \in G$. We shall show that s is an isometry. Let us consider the four vectors ξ^a, η^a, $\dot{u}^a$, ω^a, which are linearly independent. Under s they are transformed into $-\xi^a$, $-\eta^a$, $\dot{u}^a$, ω^a. It is easy to see that all scalar products between two of these vectors are left invariant by s, because $\dot{u}^a$ and ω^a are orthogonal to the trajectories. Therefore s is an isometry, and theorem 4 yields the orthogonal decomposition. As in theorem 5 we can prove that u^a is rigid.

If we apply this theorem to a non-geodesic ($\dot{u}^a \neq 0$) isometric ($u^a \sim \xi^a$) congruence the condition "$\dot{u}^a$ orthogonal to the trajectories" is automatically satisfied, and the only remaining conditions are that ω^a is orthogonal to the trajectories and not collinear to $\dot{u}^a$. This shows: instead of the two conditions (20) for the existence of orthogonal surfaces given by Kundt and Trümper [3] it is sufficient to assume their second one.

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